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MAKING THE MOST OF

STORY PROBLEMS

Honoring students' solution approaches helps teachers capitalize on the power of story problems. No more elusive train scenarios!

By Victoria R. Jacobs and Rebecca C. Ambrose

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Story problems are an important component of the mathematics curriculum, yet many adults shudder to remember their own experiences with them, often recalling the elusive train problems from high school algebra. In contrast, research shows that story problems can be powerful tools for engaging young children in mathematics, and many students enjoy making sense of these situations (NCTM 2000; NRC 2001). Honoring children's story problem approaches is of critical importance so that they construct strategies that make sense to them rather than parrot strategies they do not understand.

To explore how teachers can capitalize on the power of story problems, we chose to study teacher-student conversations in problem-solving interviews in which a K–3 teacher worked one-on-one with a child. The skills needed for productive interviewing are the same as those needed in the classroom: Teachers must observe, listen, question, design follow-up tasks, and so on. We focused our investigation on interviews because interviews isolate these important teacher-student conversations from other aspects of classroom life.

Supporting and Extending Mathematical Thinking

After analyzing videotaped problem-solving interviews conducted by 65 teachers interviewing 231 children solving 1,018 story problems, we identified eight categories of teacher moves (i.e., intentional actions) that, when timed properly, were productive in advancing mathematical conversations. We separately considered (a) the supporting moves that a teacher used before a student arrived at a correct answer and (b) the extending moves that a teacher used after the child gave a correct answer. We want to be clear that the eight categories of teacher moves we present are not intended to be a checklist that a teacher executes on every problem. Instead, we consider these moves to be a toolbox from which a teacher can draw, after considering the specific situation and instructional goals. In the midst of instruction, the most effective moves arise in response to what a child says or does and, therefore, cannot be preplanned. Because strategically responding to children's mathematical thinking is challenging, we identified our eight categories of teacher moves in an effort to assist teachers in this decision making.

Before a correct answer is given

When a child struggles or has the wrong answer, a teacher must determine how and when to intervene in order to facilitate moving the child forward without taking over the child's thinking. Supporting a student's mathematical thinking requires the teacher to "enter the child's mind" (Ginsburg 1997) as much as possible to determine what the source of difficulty might be. Then the teacher's hypotheses about a child's thinking should drive the choices made. Because "entering the child's mind" can be quite difficult, a teacher needs to be flexible and prepared to explore various supportive approaches. In our analysis, we identified four categories of moves that teachers regularly used to support a child's thinking before the student arrived at a correct answer (see **table 1**).

Table 1

Teacher Moves to Support a Child's Thinking before a Correct Answer Is Given

Category	Sample Teacher Moves
Ensure that the child understands the problem.	Ask him to explain what he knows about the problem. Rephrase or elaborate the problem. Use a more familiar or personalized context in the problem.
Change the mathematics in the problem to match the child's level of understanding.	Change the problem to use easier numbers. Change the problem to use an easier mathematical structure.
Explore what the child has already done.	Ask him to explain a partial or incorrect strategy. Ask specific questions to explore how what he has already done relates to the quantities and relationships in the problem.
Remind the child to use other strategies.	Ask him to consider using a different tool. Ask him to consider using a different strategy. Remind him of relevant strategies he has used before.

Ensure that the child understands the problem. A teacher can provide support by helping a child develop an understanding of the problem to be solved. Typical teacher moves include rereading a problem multiple times and asking a child about specific quantities in a problem (e.g., "How many puppies are in the park?"). A twist on this repetition is to ask children to explain problems in their own words. In listening to them describe a story problem in its entirety, a teacher can pinpoint what children do and do not understand.

Rephrasing or elaborating on a story can also help to engage a child. Often, this elaboration involves using a more familiar context or personalization so that the child and her friends are characters in the story. For example, a kindergartner was asked to solve the following problem:

> The teacher has twelve pencils and three baskets. If she wants to put the same number of pencils in each basket, how many pencils should she put in each basket?

The child made a pile of fifteen cubes and kept rearranging them. In response, the teacher, Mr. Reynolds, decided to elaborate and personalize the problem by involving their classroom and making himself the teacher in the story:

A teacher's most effective teaching arises in response to what a child says or does.

> Let me change it a little bit. Let's try this. Mr. Reynolds has three baskets. I have three baskets, and I have twelve pencils in my hand, and I say, "I've got to do something with these pencils. I can't walk around with them all day! What am I going to do with these pencils? Oh, here's what I'll do. I'll put some in each basket so the kids can come get them." But then I think, "I'd better put the same number in each basket. Because if I put, like, two in one basket and ten in one basket, that's not fair. So I have to put the same number of pencils in each basket." How many pencils would I put in each one of those baskets so that all the baskets would have the same number of pencils inside?

This elaborated story did not change the mathematical structure of the problem but did make the problem more real for the child, and in this case, she solved the problem correctly by using trial and error to create three piles of four cubes each. Elaborating a story may seem counterintuitive because it goes against the traditional approach of helping children identify keywords or irrelevant information in story problems. However, when elaboration is designed to make a problem more meaningful, children are more likely to avoid mechanical problem-solving approaches and instead work to make sense of the problem situation.

Change the mathematics to match the child's level of understanding. When children do not understand a problem, even after attempts to rephrase or elaborate it, changing the problem itself can be productive. One type of change is to use easier numbers. Specifically, using smaller or friendlier numbers (e.g., decade numbers) can help them gain access to the mathematics underlying a problem. After making sense of an easier problem, students generally gain confidence and, in many cases, can then make sense of the original problem.

Similarly, because research shows that children have more difficulty with some problem structures than others, another type of change is to use an easier mathematical structure (Carpenter et al. 1999). For example, a first grader was asked to solve this problem:

> Twelve mice live in a house. Nine live upstairs. How many live downstairs?

Because part-whole problems such as this do not have an explicit joining or separating action, children often do not know how the quantities relate. This student made a set of nine cubes and a set of twelve cubes and joined them to get twenty-one. After several unsuccessful attempts to help the child understand the problem, the teacher chose to change the problem to include an explicit separating action. Specifically, the teacher explained, "Nine of those mice are going to go upstairs and watch TV." In response, the girl separated nine mice from her set of twelve, leaving a group of three. This change in mathematical structure did more than allow the student to solve a problem correctly. By providing her access to an easier but related problem, the teacher created opportunities for discussing the quantities and relationships in both problems. Thus, with further skilled questioning, the teacher could use the child's understanding of the second problem to help her understand the original problem and, more generally, problems with a part-whole structure.

Explore what the child has already done. When struggling with a problem, children can sometimes determine what went wrong if they are encouraged to articulate partial or incorrect strategies. General questions, such as "Can you tell me how you solved it" or "What did you do first?" can be helpful for starting conversations, but follow-up questions require a teacher to ask about the details of a child's strategy and thus cannot be preplanned. For example, a first grader was asked to solve this problem:

> One cat has four legs. How many legs do seven cats have?

The child (C) put out seven teddy-bear counters. He saw teddy-bear counters as having two legs and two arms and, therefore, counted only two legs on each teddy bear, answering ”Fourteen.” The teacher (T) recognized that his confusion was linked to the counters he had chosen, and she posed questions to clarify how his work related to the problem context:

T: How many legs on a bear?
C: Two.
T: How many legs on a cat?
C: Four.
T: How many did you count? How many legs *each* did you count?
C: Two.
T: Is that how many legs cats have?
C: No, cats have four, and bears have two.
T: OK, could you do that again for me?
C: First I get one cat [*puts out one teddy-bear counter*], and then I get a bear [*puts out another teddy-bear counter*], and this cat has four legs, and the bear has two legs.
T: Are there bears in the story?
C: No, there's cats.

This dialogue continued for some time before the child solved the problem correctly by counting four legs on each bear and then again by using a different tool. The support the teacher provided began with what the child had already done, and through specific questioning, she helped him make sense of how his initial strategy was related (and not related) to the problem. Note that she could not have preplanned this conversation, because it grew out of her careful observation of his way of using the teddy-bear counters.

Remind the child to use other strategies. Sometimes students get lost in a particular strategy, and instead of abandoning that strategy for a more effective one, they persist in using it in unproductive ways. A teacher can help by nudging them to think more flexibly and to try alternative approaches. A simple suggestion to try a different tool or a different strategy can sometimes give a child permission to move on and self-redirect. At times, a teacher may also find that suggesting a particular tool or reminding a child of strategies used in the past is beneficial. For example, a first-grade student was asked to solve the following problem:

> Let's pretend we're out at the snack tables, and four seagulls come to the snack tables. And then seven more seagulls come to the snack tables. How many seagulls are at the snack tables?

The child first counted to four, raising one finger with each count. She then put those four fingers down. Next, she counted to seven, raising one finger with each count. At this point, the child was baffled, staring at her fingers. The teacher suggested, "Want to try it with cubes?" The child immediately made a stack of four Unifix cubes and a stack of seven Unifix cubes and then counted them altogether to get an answer of eleven. She was confident and efficient once she started using the Unifix cubes. The teacher did not tell the child how to solve the problem but did encourage her to consider using a tool that was more conducive to representing both sets; the child did not have enough fingers to show seven and four at the same time. This support reflected the teacher's understanding of children's

Table 2

Teacher Moves to Extend a Child's Thinking after a Correct Answer Is Given

Category	Sample teacher moves
Promote reflection on the strategy the child just completed.	Ask her to explain her strategy. Ask specific questions to clarify how the details of her strategy are connected to the quantities and mathematical relationships in the problem.
Encourage the child to explore multiple strategies and their connections.	Ask her to try any second strategy. Ask her to try a second strategy connected to her initial strategy in deliberate ways (e.g., more efficient counting or abstraction of work with manipulatives). Ask her to compare and contrast strategies.
Connect the child's thinking to symbolic notation.	Ask her to write a number sentence that "goes with" the problem. Ask her to record her strategy.
Generate follow-up problems linked to the problem the child just completed.	Ask her to solve the same or a similar problem with numbers that are more challenging. Ask her to solve the same or a similar problem with numbers that are strategically selected to promote more sophisticated strategies.

direct-modeling strategies in which they represent both sets before combining them.

After a correct answer is given

Solving a story problem correctly using a valid strategy is an important mathematical endeavor. However, we view problem solving as a context for having mathematical conversations, and this conversation need not end when the correct answer is reached. Instead, a teacher can pose additional questions to help students deepen their understanding of completed work and connect it to other mathematical ideas. We have identified four categories of moves that teachers regularly used to extend children's thinking after arrival at a correct answer (see **table 2**).

Promote reflection on the strategy just completed. Once a student has correctly solved a problem, a teacher can ask for a strategy explanation or for clarification about how the use of a particular strategy makes sense with the quantities and mathematical relationships expressed in the problem. Articulating these ideas can reinforce a child's understanding and give a teacher a window into that understanding. Again, attention to detail matters. Similar to the supporting questions intended to explore children's partial or incorrect strategies, teachers' extending questions were most productive when they were specific and in response to the details of what a student had already said or done. For example, a second grader was asked to solve this problem:

> This morning I had some candy. Then I gave you five pieces of candy. Now I have six pieces of candy left. How many pieces of candy did I have this morning before I gave some to you?

The student quickly solved this problem mentally and explained, "Five plus five, if you took one away, is ten and then one more is eleven, so you had eleven." Children often provide correct answers to problems with this structure, in which the initial quantity is unknown, without really understanding what they are finding. In this case, the teacher probed the child's thinking in relation to this issue:

T: So how did you know to add them together?
C: I don't know. I just added them, I guess.
T: Well, think about it. Why does that make sense for the problem?

The child thought about this question for some time and eventually used Unifix cubes to act out the story and convince himself (and the teacher) that eleven was the correct answer *and* made sense with the story. By asking him to reflect further on his strategy, the teacher ensured that he was making sense of the mathematics.

Encourage the child to explore multiple strategies and their connections. Children need opportunities to not only solve problems but also explore the mathematical connections among multiple strategies for the same problem. One approach is to ask them to generate a second strategy—any strategy—to a problem they have already solved. Another approach is to ask for a second strategy that is connected to their initial strategy in deliberate ways. For instance, a third grader using base-ten blocks to represent 12 pages of 10 spelling words per page put out 12 tens rods but counted all 120 blocks by ones! The teacher built on this initial strategy by asking her to count the blocks another way. The child responded by counting by tens and even shared that this second strategy was easier.

Another way a teacher can deliberately build on an initial strategy is to ask for a mental strategy that is an abstraction of work with manipulatives. For example, a third grader was asked to solve the following problem:

Although a teacher may initially support a student's step-by-step recording of a strategy, he should diligently support the *child's* thinking.

There are 247 girls on the playground and there are 138 boys on the playground. How many children are on the playground?

The student initially represented both quantities with base-ten flats, rods, and single cubes. Next he combined the hundred flats (3), combined the ten rods (7), combined some of the single cubes to make 10, traded the 10 single cubes for 1 tens rod, making a total of 8 tens rods, and finally counted the remaining single cubes (5) to answer 385. The teacher then asked, "Doing just what you did with the materials, could you solve that problem in your head?" The child looked at the numbers and abstracted what he had just done with the cubes. Specifically, he explained that he could add 100 to 200 to get 300 and then add 30 to 40 to get 70. Next he put 2 from the 7 with the 8 to get another 10, which made 80, and had 5 ones left, so the answer was 385. When executing this mental strategy, the child articulated the underlying mathematical idea of both strategies: combine like units and, when necessary, regroup (i.e., decompose the 7 into 5 and 2 so that the 2 can be combined with the 8 to make a new 10).

Through experiences with multiple strategies, children can gain the ability and flexibility to change strategies when one is unsuccessful. A teacher can also use multiple strategies to highlight underlying mathematical ideas by asking students to explicitly compare and contrast strategies. At times, a teacher may even ask a child to compare a successful strategy to a previously unsuccessful attempt, because, in many cases, the child will discover the reason the strategy failed.

Connect the child's thinking to symbolic notation. When solving a story problem by drawing, using manipulatives, or computing mentally, students may not use any symbolic notation. A teacher can encourage students to connect their work with mathematical symbols by asking them to either generate a number sentence that "goes with" the problem or record the strategy used to solve the problem. Although requesting a number sentence that "goes with" the problem is perhaps the more typical request, asking for a strategy representation can be powerful. Young children often begin recording their strategies in unconventional ways that include a mix of symbols and drawings. They might draw pictures of manipulatives they used and then add number labels to parts of those pictures. Over time, children's recordings become progressively more abstract until they are completely symbolic.

Generating a symbolic representation of a strategy can help children develop meaning for, and facility with, mathematical symbols because the representation is linked with their interpretation of the problem. For example, a second grader solved the problem about the number of legs on seven cats by first putting out seven tiles (cats). Next he moved two tiles to the side and said, "Four plus four equals eight." He then moved another tile to the side and said, "Eight plus four equals twelve." He continued moving one tile at a time until he had used them all, each time adding four more to his running total. When asked to write a number sentence to show what he had done, he wrote the following:

$$4 + 4 \rightarrow 8 + 4 \rightarrow 12 + 4 \rightarrow 16 + 4 \rightarrow 20 + 4 \rightarrow 24 + 4 \rightarrow 28$$

Unlike the number sentence that '"goes with" the problem ($7 \times 4 = 28$), his symbolic representation reflects how this student thought about and solved the problem. Note that his use of arrows instead of equal signs avoids the incorrect use of the equal sign between expressions.

Requesting links between strategies and symbolic notation is important so that children see the mathematics done on paper as connected to solving story problems. Moreover, once children become facile with symbolic notation, the notation itself can become a tool for problem solving and reflection. We offer a final note of caution: A teacher may initially need to support a student in recording each step of a strategy so that parts are not omitted. However, a teacher needs to be vigilant in providing support to record the child's—not the teacher's—ways of thinking about a problem.

Generate follow-up problems. By carefully sequencing problems, a teacher can create unique opportunities for mathematical discussions. Although we recognize the importance of practice, we are suggesting something beyond simply assigning additional problems to solve. We advocate that, in the midst of instruction, a teacher can consider a child's existing understanding and then modify the initial problem or create a new problem to add challenge or to encourage use of more sophisticated strategies. A first grader was asked to solve this problem:

The Kumyeey woman was collecting acorns. She had nine baskets, and she put ten acorns in each basket. So how many acorns did she have altogether?

The child quickly responded, "Ninety," explaining that he had counted, "Ten, one," putting up ten fingers and then one finger. He continued, "Twenty, two," again putting up ten fingers and this time, two fingers. He continued with this pattern of counting and finger use: "Thirty, three; forty, four; fifty, five; sixty, six; seventy, seven; eighty, eight; ninety, nine." The teacher then decided to extend the child's use of ten by posing a related problem and asking him to consider the connections:

T: So that's how you got ninety. What if she had nine baskets, but she put eleven in each basket instead of ten? [*Child thinks for a while*.] Could you use some of the work that you've already done—that we did during the afternoon—or would you have to start all over again? She still has nine baskets, and there are still ten acorns in each basket, and then she puts in one more so that each basket has eleven.

C: Ohhhhh! I get it. Well, there's already ten in each basket, so that's ninety. So I count up nine, one more nine. I mean nine ones. I'm going to add nine ones. So there's already ninety, so ninety-one, ninety-two, ninety-three, ninety-four, ninety-five, ninety-six, ninety-seven, ninety-eight, ninety-nine.

By strategically selecting numbers and by drawing attention to the link between problems, the teacher was able to further this child's base-ten understanding by helping him recognize and use the ten in the number eleven.

Summary

Our project builds on previous work on teacher questioning (see, for example, Mewborn and Huberty 1999; Stenmark 1991), which provide lists of potential questions. These lists can be important starting points for eliciting a student's thinking, but we hope that our eight categories of teacher moves (four designed to support children's thinking and four to extend it) can help teachers further customize questioning to make the most of story problems. These moves do not always lead to correct answers, and we reiterate that not all eight are intended to be used in every situation. However, together they form a toolbox from which teachers can select means to help students solve problems and explore connections among mathematical ideas. Engaging in mathematical discussions about story problems is challenging; we offer three final guidelines:

- **Elicit and respond to a child's ideas.** The most effective teacher moves cannot be preplanned. Instead, they must occur in response to a student's specific actions or ideas. Thus, expertise is tied less to planning before a student arrives and more to seeding conversations, finding the mathematics in children's comments and actions, and making in-the-moment decisions about how to support and extend children's thinking.
- **Attend to details in a child's strategy and talk.** Research on children's developmental trajectories shows that subtle differences in children's strategies and talk can reflect important distinctions in their mathematical understandings (NRC 2001). A teacher can customize instruction on the basis of these distinctions, and by attending to details of a child's explanations and comments, a teacher also communicates respect for a child's ideas.
- **Do not always end a conversation after a correct answer is given.** Important learning can occur after students give a correct answer if the teacher asks them to articulate, reflect on, and build on their initial strategies.

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